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An analogy between the spin-2 and superspin-$\frac{3}{2}$ equations of motion

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Abstract. The general schema for obtaining covariant wave equations is given. The structure of possible equations for a symmetrical tensor field describing spin-2 and -0 is examined. The general form of superfield equations of motion is proposed. The structure of possible equations for vector and scalar superfields describing superspin-$\frac{3}{2}$ and -0 is examined. The derivation of equations for a symmetrical tensor field is analogous to the derivation of superfield equations for vector and scalar superfields.

1. Introduction

The development of supergravity (van Nieuwenhuizen 1981) points out the importance of spin-$\frac{3}{2}$ amd spin-2 particles. For that reason it is useful to investigate the possible description of spin-$\frac{3}{2}$ and spin-2 states more thoroughly. In our previous paper (Loide 1984), the full description of equations for a vector-bispinor $\psi_{\alpha\mu}$ is given. A vector-bispinor allows us to describe spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ states in massive and massless cases. In the case of spin-2 state, a symmetrical tensor $h^{\mu\nu}$ is mostly used. In this paper we give the full description of second-order wave equations for a symmetrical tensor $h^{\mu\nu}$.

The supersymmetry introduces more complicated objects—superfields (Salam and Strathdee 1974), which contain ordinary Bose and Fermi fields. The irreducible multiplet with a superspin Y contains in the massive case the following Poincaré spins—$s = Y-\frac{1}{2}, Y, Y, Y+\frac{1}{2}$. Since the superfields are reducible, similar equations of motion as in the ordinary field case are needed. The role of superfield equations of motion is therefore to describe a given superspin Y with a given mass m, or several superspins with given mass spectra. The full theory of superfield equations of motion has not yet been worked out. It seems that the principles that are used in the ordinary wave equations case work also in the superfield case. Ogievetsky and Sokatchev (1976, 1977a, b) generalised the root method into the superfield case. However the root method is not the most general one and there are superfield equations of motion which are not derivable via the root method (Loide and Suurvarik 1983). Here we show that the method of spin projection operators used in the ordinary field case may be generalised into the superfield case. As an example, we give the superfield equation for vector and scalar superfields, which describes single superspin-$\frac{3}{2}$ or superspins-$\frac{3}{2}$ and -0. Superspin-$\frac{3}{2}$ was previously described with the help of the vector superfield $h^{\mu}(x, \theta)$ (Ogievetsky and Sokatchev 1976, 1977a, b), but this equation describes also superspin-0 states having unphysical mass. Scalar superfield is introduced to eliminate superspin-0 states or to give them physical masses. We also demonstrate that the derivation of equations for a symmetrical tensor field $h^{\mu\nu}(x)$ is entirely analogous to the derivation of superfield equations for a vector and scalar superfields $h^{\mu}(x, \theta)$ and $\phi(x, \theta)$.

The paper is organised as follows. In § 2, the general formalism of spin projection operators is introduced. In §§ 3, 4 and 5, equations for a symmetrical tensor $h^{\mu\nu}(x)$ are studied. The general form of superfield equations of motion is introduced in § 6 and, in § 7, the case of vector and scalar superfields is illustrated. In the appendix, the superfield projection operators are given.

2. General formalism of spin-projection operators

At the beginning we briefly discuss the construction of wave equations, using the formalism of spin-projection operators. The basic principles were recently given in Loide (1984). Here we add the covariant form of equations.

Let us deal with the nth-order equation

$$\mathrm{i}\,\partial_{\mu_1}\ldots\mathrm{i}\,\partial_{\mu_n}\beta^{\mu_1\ldots\mu_n}\psi(x)=m^n\psi(x), \tag{2.1}$$

where the matrices $\beta^{\mu_1\ldots\mu_n}$ satisfy the following commutation relations

$$[S^{\mu\nu},\beta^{\mu_1\ldots\mu_n}]=\sum\eta^{\nu\mu_l}\beta^{\mu_1\ldots\mu\ldots\mu_n}-\eta^{\mu\mu_l}\beta^{\mu_1\ldots\nu\ldots\mu_n}, \tag{2.2}$$

and $\eta^{\mu\nu}=\mathrm{diag}(+---)$. The generators of the Lorentz group $S^{\mu\nu}$ satisfy

$$[S^{\mu\nu},S^{\rho\sigma}]=\eta^{\nu\rho}S^{\mu\sigma}+\eta^{\mu\sigma}S^{\nu\rho}-\eta^{\mu\rho}S^{\nu\sigma}-\eta^{\nu\sigma}S^{\mu\rho}. \tag{2.3}$$

In spite of the fact that only the first- and second-order equations are usually used, since the higher-order equations give the mass spectrum where unphysical masses are present, the higher-order equations appear in some recently used methods. The root method (Ogievetsky and Sokatchev 1976, 1977b, Berends *et al* 1979, Berends and van Reisen 1980), for example, also operates with higher-order equations.

In the non-covariant form, where the mass and spin spectrum is mostly analysed, the problem reduces to the derivation of $\beta^{0\ldots0}$ matrix. In the representation where ψ is decomposed into the direct sum of irreducible representations $i=(k_i, l_i)$: $1\oplus2\oplus\ldots\oplus r$, $\beta^{0\ldots0}$ has a general form (Loide 1972)

$$\beta^{0\ldots0}=\begin{vmatrix} a_{11}t_{11} & a_{12}t_{12} & \ldots & a_{1r}t_{1r}\\ a_{21}t_{21} & a_{22}t_{22} & \ldots & a_{2r}t_{2r}\\ \vdots & \vdots & & \vdots\\ a_{n1}t_{n1} & a_{r2}t_{r2} & \ldots & a_{rr}t_{rr}\end{vmatrix},\qquad \psi=\begin{vmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_r\end{vmatrix}, \tag{2.4}$$

where a_{ij} are arbitrary free parameters. The matrices t_{ij} are expressed with the help of spin-projection operators t^s_{ij} as

$$t_{ij}=\sum_s\alpha_{ij}(s)t^s_{ij}, \tag{2.5}$$

where the summation is over all common spins in representations i and j.

The spin-projection operators t^s_{ij} are expressed with the help of Clebsch–Gordan coefficients

$$t^s_{ij}=\sum_\sigma\langle(i)|s\sigma\rangle\langle s\sigma|(j)\rangle \tag{2.6}$$

and satisfy

$$t^s_{ij} t^{s'}_{jk} = \delta_{ss'} t^s_{ik} \tag{2.7}$$

(there is no summation over j).

The coefficients $\alpha_{ij}(s)$ in (2.5) are, in general, not arbitrary but depend on the order of equation n, on the representations i and j, and on the spin s. The general relations for $\alpha_{ij}(s)$ were given in Loide (1972). In the case of first-order equations these coefficients are well known (see, e.g. Loide 1984a).

Now we shall discuss how the covariant form (2.1) is expressed with the help of matrix $\beta^{0\ldots0}$. Instead of t^s_{ij}, one must derive covariant spin-projection operators P^s_{ij} which satisfy the same relations as t^s_{ij}

$$P^s_{ij} P^{s'}_{jk} = \delta_{ss'} P^s_{ik}. \tag{2.8}$$

Here we exploit the covariant operators Π^s_{ij} previously used by Weinberg (1964a, b, 1969), Pursey (1965) and Tung (1966, 1967)

$$\Pi^s_{ij} = \partial_{\mu_1} \ldots \partial_{\mu_l} t^{s\mu_1\ldots\mu_l}_{ij}, \tag{2.9}$$

where $t^{s0\ldots0}_{ij} = t^s_{ij}$ and

$$l = 2\min\{(k_i + k_j), (l_i + l_j)\}. \tag{2.10}$$

If we define non-local operators P^s_{ij}

$$P^s_{ij} = (\Box)^{-l/2} \Pi^s_{ij}, \tag{2.11}$$

$\Box = \partial_\mu \partial^\mu$, the operators P^s_{ij} satisfy the required relations (2.8). From (2.10), P^s_{ij} contain non-local terms depending on the representations i and j up to $(\Box)^{-l/2}$.

The covariant form (2.1) is expressed in a different but equivalent form

$$(-\Box)^{n/2} \beta^{0\ldots0}(\partial) \psi(x) = m^n \psi(x), \tag{2.12}$$

where $\beta^{0\ldots0}(\partial)$ is

$$\beta^{0\ldots0}(\partial) = \begin{vmatrix} a_{11}P_{11} & a_{12}P_{12} & \ldots & a_{1r}P_{1r} \\ a_{21}P_{21} & a_{22}P_{22} & \ldots & a_{2r}P_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1}P_{r1} & a_{r2}P_{r2} & \ldots & a_{rr}P_{rr} \end{vmatrix} \tag{2.13}$$

and

$$P_{ij} = \sum_s \alpha_{ij}(s) P^s_{ij}. \tag{2.14}$$

Comparing (2.13), (2.14) with (2.4), (2.5), it is easy to see that one must change $t^s_{ij} \to P^s_{ij}$. When the coefficients $\alpha_{ij}(s)$ are not known, then the following remark is useful—$\alpha_{ij}(s)$ must be so chosen that the non-local terms in P_{ij} are not higher than $(\Box)^{-n/2}$.

In § 6 we generalise the last form (2.12)–(2.14) into the superfield case where one must find the corresponding superspin projection operators E^y_{ij}.

The mass and spin spectrum is determined with the help of reduced matrices β_s formed from the coefficients $a_{ij}\alpha_{ij}(s)$

$$\beta_s = \begin{vmatrix} a_{11}\alpha_{11}(s) & a_{12}\alpha_{12}(s) & \dots & a_{1r}\alpha_{1r}(s) \\ a_{21}\alpha_{21}(s) & a_{22}\alpha_{22}(s) & \dots & a_{2r}\alpha_{2r}(s) \\ \vdots & \vdots & & \vdots \\ a_{r1}\alpha_{r1}(s) & a_{r2}\alpha_{r2}(s) & \dots & a_{rr}\alpha_{rr}(s) \end{vmatrix}. \tag{2.15}$$

If β_s has non-zero eigenvalues λ_i, then the masses corresponding to a given s are $m(\lambda_i)^{-1/n}$. In the case of first-order equations the non-zero eigenvalues are $\pm\lambda_i$ and the corresponding mass is $m\lambda_i^{-1}$. In the case of second-order equations, the mass corresponding to a non-zero eigenvalue λ_i is $m\lambda_i^{-1/2}$.

3. Equations for $h^{\mu\nu}$

In this section we shall illustrate the general method given in § 2 for the case of second-order spin-2 equations for a symmetric tensor $h^{\mu\nu}$ ($h^{\mu\nu} = h^{\nu\mu}$). $h^{\mu\nu}$ is represented as a direct sum of two irreducible representations $1 = (1, 1)$: $H^{\mu\nu} = h^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}h^{\rho}{}_{\rho}$ and $2 = (0, 0)$: $h = \frac{1}{4}h^{\rho}{}_{\rho}$.

We shall start to deal with the representation 1. Covariant spin-projection operators P^s_{11} are calculated with the help of covariant spin operator $S^2 = \frac{1}{2}(S^{\mu\nu}S_{\nu\mu} + (2\partial_\mu\,\partial^\nu/\Box)S^{\mu\rho}S_{\nu\rho})$

$$\begin{aligned}
(P^2_{11})^{\mu\nu}{}_{\kappa\lambda} &= \tfrac{1}{2}[\eta^{\mu}{}_{\kappa}\eta^{\nu}{}_{\lambda} + \eta^{\mu}{}_{\lambda}\eta^{\nu}{}_{\kappa} - \tfrac{2}{3}\eta^{\mu\nu}\eta_{\kappa\lambda} - (\partial^\mu\,\partial_\kappa/\Box)\eta^{\nu}{}_{\lambda} - (\partial^\mu\,\partial_\lambda/\Box)\eta^{\nu}{}_{\kappa} \\
&\quad - (\partial^\nu\,\partial_\kappa/\Box)\eta^{\mu}{}_{\lambda} - (\partial^\nu\,\partial_\lambda/\Box)\eta^{\mu}{}_{\kappa} + \tfrac{2}{3}(\partial^\mu\,\partial^\nu/\Box)\eta_{\kappa\lambda} \\
&\quad + \tfrac{2}{3}(\partial_\kappa\,\partial_\lambda/\Box)\eta^{\mu\nu} + \tfrac{4}{3}(\partial^\mu\,\partial^\nu\,\partial_\kappa\,\partial_\lambda/\Box^2)] \\
(P^1_{11})^{\mu\nu}{}_{\kappa\lambda} &= \tfrac{1}{2}[(\partial^\mu\,\partial_\kappa/\Box)\eta^{\nu}{}_{\lambda} + (\partial^\mu\,\partial_\lambda/\Box)\eta^{\nu}{}_{\kappa} \\
&\quad + (\partial^\nu\,\partial_\kappa/\Box)\eta^{\mu}{}_{\lambda} + (\partial^\nu\,\partial_\lambda/\Box)\eta^{\mu}{}_{\kappa} - 4(\partial^\mu\,\partial^\nu\,\partial_\kappa\,\partial_\lambda/\Box^2)] \\
(P^0_{11})^{\mu\nu}{}_{\kappa\lambda} &= \tfrac{1}{3}[\tfrac{1}{4}\eta^{\mu\nu}\eta_{\kappa\lambda} - (\partial^\mu\,\partial^\nu/\Box)\eta_{\kappa\lambda} - (\partial_\kappa\,\partial_\lambda/\Box)\eta^{\mu\nu} + 4(\partial^\mu\,\partial^\nu\,\partial_\kappa\,\partial_\lambda/\Box^2)]
\end{aligned} \tag{3.1}$$

P^s_{11} are non-local and contain terms $\Box^{-1}$, $\Box^{-2}$.

P_{11} has a general form

$$P_{11} = \alpha(2)P^2_{11} + \alpha(1)P^1_{11} + \alpha(0)P^0_{11} \tag{3.2}$$

where $\alpha(s)$ satisfy (Loide 1972)

$$\tfrac{1}{6}\alpha(2) - \tfrac{1}{2}\alpha(1) + \tfrac{1}{3}\alpha(0) = 0. \tag{3.3}$$

As we have noted in § 2, when the coefficients $\alpha(s)$ are not known, they must be so chosen that in the case of the second-order equation P_{11} has non-local terms up to $\Box^{-1}$. It is easy to verify that $\alpha(s)$ which satisfy (3.3) eliminate non-local terms $\Box^{-2}$.

Due to the relation (3.3), all three spins may be present in our equation or one of them may be eliminated. If we want to describe spin-2, then $\alpha(2) \neq 0$, but $\alpha(1)$ or $\alpha(0)$ may be chosen equal to zero. Usually spin-1 is eliminated and we set $\alpha(1) = 0$. We choose $\alpha(2) = 1$, then $\alpha(0) = -\frac{1}{2}$ and P_{11} has a form

$$P_{11} = P^2_{11} - \tfrac{1}{2}P^0_{11}. \tag{3.4}$$

Recently the higher-order representations are used to describe lower spins (Deser *et al* 1981, Cox 1982). Using $h^{\mu\nu}$, it is possible to write down an equation which describes spin-1, or spin-1 and spin-0 particles. In that case one must set $\alpha(2)=0$, $\alpha(1)=1$ and $\alpha(0)=\frac{3}{2}$.

From (3.4) it is obvious why the scalar representation $2=(0,0)$ is needed. If we write an equation $-\Box P_{11}\psi_1=m^2\psi_1$, spin-0 is present and has non-physical mass $i\sqrt{2}m$. The representation 2 is therefore used to eliminate spin-0 states, or to give to spin-0 states physical mass. In the case of scalar representation, P^0_{22} is

$$(P^0_{22})^{\mu\nu}{}_{\kappa\lambda}=\tfrac{1}{4}\eta^{\mu\nu}\eta_{\kappa\lambda}. \tag{3.5}$$

The covariant operators P^0_{12} and P^0_{21} which satisfy (2.8) are the following

$$\begin{aligned}(P^0_{12})^{\mu\nu}{}_{\kappa\lambda}&=\tfrac{1}{12}\sqrt{3}[\eta^{\mu\nu}-4(\partial^\mu\,\partial^\nu/\Box)]\eta_{\kappa\lambda}\\(P^0_{21})^{\mu\nu}{}_{\kappa\lambda}&=\tfrac{1}{12}\sqrt{3}\eta^{\mu\nu}(\eta_{\kappa\lambda}-4\,\partial_\kappa\,\partial_\lambda/\Box).\end{aligned}\tag{3.6}$$

The general covariant equation is obtained as follows. We choose $a_{11}=1$ (then $s=2$ state has mass m) and denote $a_{21}=a$, $a_{12}=b$, $a_{22}=c$. Then (2.12) takes the form

$$-\Box\begin{vmatrix}P^2_{11}-\frac{1}{2}P^0_{11} & bP^0_{12}\\ aP^0_{21} & cP^0_{22}\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}=m^2\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}. \tag{3.7}$$

Using the expressions of operators P^s_{ij}, ψ_1 and ψ_2, we have from (3.7)

$$\begin{aligned}&\Box h^{\mu\nu}-\partial^\mu\,\partial_\kappa h^{\kappa\nu}-\partial^\nu\,\partial_\kappa h^{\kappa\mu}+\tfrac{1}{2}\eta^{\mu\nu}\,\partial_\kappa\,\partial_\lambda h^{\kappa\lambda}+(\tfrac{1}{2}-\tfrac{1}{3}\sqrt{3}b)\,\partial^\mu\,\partial^\nu h^\rho{}_\rho\\&\qquad+\tfrac{1}{8}(\tfrac{2}{3}\sqrt{3}b-3)\eta^{\mu\nu}\Box h^\rho{}_\rho+m^2(h^{\mu\nu}-\tfrac{1}{4}\eta^{\mu\nu}h^\rho{}_\rho)=0,\\&\eta^{\mu\nu}[-\tfrac{1}{3}\sqrt{3}a\,\partial_\kappa\,\partial_\lambda h^{\kappa\lambda}+\tfrac{1}{4}(\tfrac{1}{3}\sqrt{3}a+c)\Box h^\rho{}_\rho+m^2\tfrac{1}{4}h^\rho{}_\rho]=0.\end{aligned}$$

The last system may be written as an equation for symmetrical tensor $h^{\mu\nu}$. If we add these two equations, we obtain

$$\begin{aligned}&\Box h^{\mu\nu}-\partial^\mu\,\partial_\kappa h^{\kappa\nu}-\partial^\nu\,\partial_\kappa h^{\kappa\mu}+(\tfrac{1}{2}-\tfrac{1}{3}\sqrt{3}a)\eta^{\mu\nu}\partial_\kappa\,\partial_\lambda h^{\kappa\lambda}\\&\qquad+(\tfrac{1}{2}-\tfrac{1}{3}\sqrt{3}b)\,\partial^\mu\,\partial^\nu h^\rho{}_\rho+\tfrac{1}{8}[\tfrac{2}{3}\sqrt{3}(a+b)+2c-3]\eta^{\mu\nu}\Box h^\rho{}_\rho+m^2h^{\mu\nu}=0.\end{aligned}\tag{3.8}$$

The most general equation for $h^{\mu\nu}$ describing spin-2 and spin-0 states has, therefore, the following general form

$$\Box h^{\mu\nu}-\partial^\mu\,\partial_\kappa h^{\kappa\nu}-\partial^\nu\,\partial_\kappa h^{\kappa\mu}+A\eta^{\mu\nu}\,\partial_\kappa\,\partial_\lambda h^{\kappa\lambda}+B\,\partial^\mu\,\partial^\nu h^\rho{}_\rho+C\eta^{\mu\nu}\Box h^\rho{}_\rho+m^2h^{\mu\nu}=0. \tag{3.9}$$

The relation between the coefficients a, b, c and A, B, C is the following

$$\begin{aligned}A&=\tfrac{1}{6}\sqrt{3}(\sqrt{3}-2a) & a&=\tfrac{1}{2}\sqrt{3}(1-2A)\\B&=\tfrac{1}{6}\sqrt{3}(\sqrt{3}-2b) & b&=\tfrac{1}{2}\sqrt{3}(1-2B)\\C&=\tfrac{1}{24}\sqrt{3}(2a+2b+2\sqrt{3}c-3\sqrt{3}) & c&=\tfrac{1}{2}(2A+2B+8C+1).\end{aligned}\tag{3.10}$$

As we have seen in this section, the ordinary procedure for obtaining relativistically invariant equations work also in the covariant case, one needs only covariant spin-projection operators P^s_{ij}. Sometimes the other method—the root method is used (Ogievetsky and Sokatchev 1976, 1977b, Berends *et al* 1979, Berends and van Reisen 1980). In principle the root method operates with the same operators P^s_{ij}, but it starts from the higher-order equation $(-\Box)^{l/2}P^s_{ii}\psi=m^l\psi$, which guarantees that the needed spin s is extracted, and then the first- or second-order equation is derived, which leads to the equation we started with. The root method allows us to obtain equations where some single-spin state is present; in the multi-particle case it is not applicable. In our method we first construct the most general equation of the required order and then the analysis of mass and spin spectrum reduces to a pure matrix-algebraic problem. In the next section we explain the latter procedure in the case of the equation for a symmetric tensor $h^{\mu\nu}$.

4. Mass spectrum

To determine the mass spectrum it is useful to decompose $\beta^{00}(\partial)=\beta^2(\partial)+\beta^0(\partial)$. From (3.7)

$$\beta^2(\partial)=\begin{vmatrix} P^2_{11} & 0 \\ 0 & 0 \end{vmatrix} \qquad \beta^0(\partial)=\begin{vmatrix} -\frac{1}{2}P^0_{11} & bP^0_{12} \\ aP^0_{21} & cP^0_{22} \end{vmatrix}. \tag{4.1}$$

The investigation of eigenvalues of matrices $\beta^2(\partial)$ and $\beta^0(\partial)$ reduces to the investigation of 2×2 matrices β_2 and β_0 formed from the coefficients $a_{ij}\alpha_{ij}(s)$

$$\beta_2=\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \qquad \beta_0=\begin{vmatrix} -\frac{1}{2} & b \\ a & c \end{vmatrix}. \tag{4.2}$$

Masses corresponding to a given spin s are determined with the help of non-zero eigenvalues λ_i of matrix β_s in the following way $m_i=m\lambda_i^{-1/2}$. β_2 has one non-zero eigenvalue $\lambda=1$, which means that the spin-2 state has mass m. The mass spectrum of spin-0 states depends on the choice of free parameters a, b and c. By a proper choice of these parameters equation (3.7) describes two, one or no spin-0 states.

Next, we give a detailed analysis of spin-0 mass spectrum. The eigenvalues of β_0 are calculated from

$$4\lambda_{1,2}=2c-1\pm[(2c+1)^2+16ab]^{1/2}. \tag{4.3}$$

As we can see, λ_1 and λ_2 depend on two free parameters ab and c. In order to have physical masses, λ_1 and λ_2 must be real and non-negative. It leads to the following restrictions

$$c\geqslant\tfrac{1}{2}, \qquad 2ab+c\leqslant 0, \qquad (2c+1)^2+16ab\geqslant 0. \tag{4.4}$$

The region determined by the relations (4.4) is called the physical region since it gives the physical mass spectrum. The physical region is determined by the line $2ab+c=0$ and parabola $(2c+1)^2+16ab=0$.

The points on the line

$$2ab+c=0 \tag{4.5}$$

give the eigenvalues

$$\lambda_1=c-\tfrac{1}{2}, \qquad \lambda_2=0. \tag{4.6}$$

This equation describes in addition to spin-2 state one spin-0 state with mass $m\lambda^{-1/2}$. β_0 satisfies $\beta_0(\beta_0-\lambda_1)=0$.

The points on the parabola

$$(2c+1)^2+16ab=0 \tag{4.7}$$

give coincident eigenvalues

$$\lambda_1 = \lambda_2 = \tfrac{1}{4}(2c-1). \tag{4.8}$$

These points must be treated as unphysical since β gives the minimal polynomial $(\beta_0 - \lambda_1)^2 = 0$. β_0 has only one eigenvector and gives vanishing energy density (see, e.g., Cox 1977).

The other points in the physical region (4.4) give two different non-zero eigenvalues λ_1 and λ_2, and describe two spin-0 states with masses $m\lambda_1^{-1/2}$ and $m\lambda_2^{-1/2}$.

If we want to describe the single spin-2 state, we must choose $\lambda_1 = \lambda_2 = 0$. From (4.6) and (4.8) we have

$$ab = -\tfrac{1}{4}, \qquad c = \tfrac{1}{2}. \tag{4.9}$$

In the case of spin-2 equation β_0 and also $\beta^0(\partial)$ are nilpotent: $(\beta_0)^2 = 0$.

As is well known (Velo and Zwanziger 1969), single particle equations have in general acausality defects when the external fields are present. In Velo (1972) it is demonstrated that acausality is present also in the spin-2 equation for $h^{\mu\nu}$. In the case of first-order equations acausality is connected with the essential nilpotency of β matrices (see, e.g., Loide 1983). In our case spin-2 equation contains also nilpotent matrices, since $\beta^0(\partial)$ is nilpotent, and for that reason it seems that in the case of second-order equations the origin of acausality is the same—it is caused by nilpotent states. In the case of first-order equations acausality is absent when the β matrices are diagonalisable (Amar and Dozzio 1975). The acausality problem for second-order equations seems more complicated (Amar *et al* 1980), and it is not known whether the equations with diagonalisable β matrices are causal in the presence of minimal electromagnetic coupling. From our results there seems no reason for the equations with diagonalisable matrices to have acausality defects, since the method of characteristics used in Velo and Zwanziger (1969) leads to the spacelike normals and causal propagation in the case of diagonalisable β matrices when $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

In our previous analysis we have solved the following problem—how to determine the mass and spin content of a given equation (3.9). In that case the coefficients A, B and C are known. From (3.10) one calculates a, b and c, and then the eigenvalues λ_1 and λ_2, which determined masses of spin-0 particles, are calculated from (4.3).

The inverse problem—how to find an equation with a given mass spectrum can also be easily solved. From the masses of spin-0 states one finds eigenvalues λ_1 and λ_2. The coefficients ab and c are obtained as follows

$$\begin{aligned} ab &= -\lambda_1\lambda_2 - \tfrac{1}{2}(\lambda_1 + \lambda_2) - \tfrac{1}{4}, \\ c &= \lambda_1 + \lambda_2 + \tfrac{1}{2}. \end{aligned} \tag{4.10}$$

5. Invariant scalar product, Lagrangian

Invariant scalar product which defines the Hermitising matrix Λ is obtained similarly as in the case of equations for a vector-bispinor (Loide 1984)

$$(h, h) \equiv h^+ \Lambda h = h^+_{\mu\nu} h^{\mu\nu} - \tfrac{1}{4}(1 - b/a) h^{+\mu}{}_{\mu} h^{\nu}{}_{\nu} \tag{5.1}$$

(we have chosen a and b to be real).

If we introduce the conjugated wavefunction

$$\tilde{h}_{\mu\nu} = h^{+}_{\rho\sigma}[\eta^{\rho}_{\mu}\eta^{\sigma}_{\nu} - \tfrac{1}{4}(1-b/a)\eta^{\rho\sigma}\eta_{\mu\nu}] \tag{5.2}$$

the equation (3.11) is obtained from the following Lagrangian

$$L = -(\partial_\rho \tilde{h})\beta^{\rho\sigma}(\partial_\sigma h) + m^2\tilde{h}h. \tag{5.3}$$

Usually L is varied with respect to $h^{+}_{\rho\sigma}$ and then the equation obtained has a mass term $m^2(\Lambda h)^{\mu\nu}$, and not $m^2h^{\mu\nu}$ as in our case. We illustrate the difference in the case of single spin-2 equation which is written in the form (Ogievetsky and Sokatchev 1976, 1977b)

$$\Box h^{\mu\nu} - \partial^\mu \partial_\kappa h^{\kappa\nu} - \partial^\nu \partial_\kappa h^{\kappa\mu} + \frac{1+\alpha}{1+2\alpha}(\eta^{\mu\nu}\partial_\kappa \partial_\lambda h^{\kappa\lambda} + \partial^\mu \partial^\nu h^{\rho}{}_{\rho})$$
$$-\frac{2+4\alpha+3\alpha^2}{2(1+2\alpha)^2}\eta^{\mu\nu}\Box h^{\rho}{}_{\rho} - \frac{1+\alpha+\alpha^2}{(1+2\alpha)^2}m^2\eta^{\mu\nu}h^{\rho}{}_{\rho} + m^2h^{\mu\nu} = 0, \tag{5.4}$$

where $\alpha \neq -\frac{1}{2}$ is an arbitrary free parameter.

Our spin-2 equation has a form ($ab = -\frac{1}{4}$, $c = \frac{1}{2}$)

$$\Box h^{\mu\nu} - \partial^\mu \partial_\kappa h^{\kappa\nu} - \partial^\nu \partial_\kappa h^{\kappa\mu} + (\tfrac{1}{2} - \tfrac{1}{3}\sqrt{3}a)\eta^{\mu\nu}\partial_\kappa \partial_\lambda h^{\kappa\lambda}$$
$$+ (\tfrac{1}{2} - \tfrac{1}{3}\sqrt{3}b)\partial^\mu \partial^\nu h^{\rho}{}_{\rho} + \tfrac{1}{8}[\tfrac{2}{3}\sqrt{3}(a+b) - 2]\eta^{\mu\nu}\Box h^{\rho}{}_{\rho} + m^2h^{\mu\nu} = 0. \tag{5.5}$$

It is easy to verify that the relation between a, b and α is the following

$$a = \tfrac{1}{6}\sqrt{3}(1+2\alpha), \qquad b = -\sqrt{3}/2(1+2\alpha), \tag{5.6}$$

and (5.5) is obtained from (5.4), multiplying it by $(\eta^{\rho}{}_{\mu}\eta^{\sigma}{}_{\nu} - \frac{1}{3}(1+\alpha+\alpha^2)\eta^{\rho\sigma}\eta_{\kappa\lambda})$.

To conclude this section, we shall briefly discuss the zero rest mass case. At the beginning we determine the coefficients A, B and C in (3.9), which in the case of $m^2 = 0$ is invariant under gauge transformations

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu h^\nu + \partial^\nu h^\mu. \tag{5.7}$$

We get $A = -C$ and $B = 1$, which gives

$$\sqrt{3}a = c, \qquad b = -\tfrac{1}{2}\sqrt{3}. \tag{5.8}$$

These points lie on the line (4.5) which, in the massive case, describes one spin-0 state. The zero mass equation is obtained if in addition to gauge transformations (5.7) we demand that $\partial_\mu(\partial_\rho \partial_\sigma \beta^{\rho\sigma})^{\mu\nu}{}_{\kappa\lambda}h^{\kappa\lambda} = 0$. It gives $A = B = -C = 1$, and the equation has a well known form (Ogievetsky and Sokatchev 1976, van Nieuwenhuizen 1981)

$$\Box h^{\mu\nu} - \partial^\mu \partial_\kappa h^{\kappa\nu} - \partial^\nu \partial_\kappa h^{\kappa\mu} + \eta^{\mu\nu}\partial_\kappa \partial_\lambda h^{\kappa\lambda} + \partial^\mu \partial^\nu h^{\rho}{}_{\rho} - \eta^{\mu\nu}\Box h^{\rho}{}_{\rho} = 0. \tag{5.9}$$

As we have seen the equation which leads to the zero mass limit is different from the equation which describes in the massive case the single spin-2 state, since these equations correspond to the different choice of coefficients a, b and c (or A, B and C). This fact explains the van Dam–Veltman theorem (van Dam and Veltman 1970) recently discussed in Berends and van Reisen (1980). The Van Dam–Veltman theorem states that for higher spins the $m \to 0$ limit of the amplitude for the exchange of a massive higher spin particle between two external sources is different from the amplitude for the exchange of a massless particle. These two amplitudes must be different since the equations in the massive and massless cases are different. From

(5.2) also the scalar product depends on the choice of coefficients a and b. The same situation takes place in the spin-$\frac{3}{2}$ case too where the massive spin-$\frac{3}{2}$ equation differs from the equation which admits zero mass limit (Loide 1984).

In this paper we have demonstrated that the ordinary method discussed in § 2 allows us to investigate the general structure of all equations with a given ψ representation. The equations which leads to the zero mass limit are in the massive case in general multi-particle equations and for that reason the investigation of all equations with a given ψ representation is useful. When it concerns the root method, frequently used (Ogievetsky and Sokatchev 1976, 1977b, Berends *et al* 1979, Berends and van Reisen 1980), it should be remarked that this method allows us to derive in general only single-particle equations.

6. Superfield equations of motion

In the superfield case analogous equations as in the ordinary field case are needed. The role of superfield equations of motion is to describe a supermultiplet with a given mass and superspin Y, or several masses and superspins.

Ogievetsky and Sokatchev (1976, 1977b) generalised the root method into the superfield case and proposed some superfield equations of motion. As in the ordinary field case, the root method is not the most universal one. There are equations which are not derivable via the root method, as, for example, the equation for chiral spinor superfield proposed by Salam and Strathdee (1975). In this section we generalise the ordinary method of § 2 into the superfield case.

The general superfield with a Lorentz index i is the following

$$\begin{aligned}\phi_i(x,\theta) = {} & A_i(x)+\bar{\theta}\psi_i(x)+\tfrac{1}{4}\bar{\theta}\theta F_i(x)+\tfrac{1}{4}\bar{\theta}\gamma^5\theta G_i(x)\\ & +\tfrac{1}{4}\bar{\theta}\mathrm{i}\gamma^\mu\gamma^5\theta A_{\mu i}(x)+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\chi_i(x)+\tfrac{1}{32}(\bar{\theta}\theta)^2 D_i(x).\end{aligned} \tag{6.1}$$

We use the following notation: θ_α is a four-component anticommuting Majorana spinor, $\bar{\theta}_\beta = C^{-1}_{\beta\alpha}\theta_\alpha$, $C=\mathrm{i}\gamma^0\gamma^2$, $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$, $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$.

Now we are able to give the general method for deriving superfield equations of motion for a superfield $\Phi(x,\theta)$. We use a representation where $\Phi(x,\theta)$ is a direct sum of superfields $\phi_i(x,\theta)$ which transform under Lorentz group according to some irreducible representation (k_i, l_i). Now we propose that there exists a set of superspin-projection operators E^y_{ij} which satisfy the relations similar to (2.8)

$$E^y_{ij}E^{y'}_{jk} = \delta_{yy'}E^y_{ik}. \tag{6.2}$$

The general nth-order superfield equation for $\Phi(x,\theta)$ may be written in the following form

$$(-\Box)^{n/2}\pi^{0\ldots0}\Phi(x,\theta) = m^n\Phi(x,\theta), \tag{6.3}$$

where $\pi^{0\ldots0}$ is

$$\pi^{0\ldots0} = \begin{vmatrix} a_{11}E_{11} & a_{12}E_{12} & \ldots & a_{1r}E_{1r}\\ a_{21}E_{21} & a_{22}E_{22} & \ldots & a_{2r}E_{2r}\\ \vdots & \vdots & & \vdots\\ a_{r1}E_{r1} & a_{r2}E_{r2} & \ldots & a_{rr}E_{rr}\end{vmatrix}, \qquad \Phi = \begin{vmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_r\end{vmatrix} \tag{6.4}$$

and E_{ij} are the following

$$E_{ij} = \sum_{y} \alpha_{ij}(Y) E^{y}_{ij}. \tag{6.5}$$

Since we have no general expressions for $\alpha_{ij}(Y)$, these coefficients must be chosen so that for a nth-order equation (6.3) maximal non-locality of all operators E_{ij} is not higher than $(\Box)^{-n/2}$.

The analysis of superspin and mass spectrum of the general equation (6.3) is the same as in the ordinary field case. One must decompose

$$\pi^{0\ldots0} = \pi^{y_1} + \pi^{y_2} + \ldots + \pi^{y_k}, \tag{6.6}$$

where π^{y} contains projection operators E^{y}_{ij} of a given superspin Y. Masses corresponding to a superspin Y are determined with the help of non-zero eigenvalues of matrix π_y formed from the coefficients $a_{ij}\alpha_{ij}(Y)$

$$\pi_y = \begin{vmatrix} a_{11}\alpha_{11}(Y) & a_{12}\alpha_{12}(Y) & \ldots & a_{1r}\alpha_{1r}(Y) \\ a_{21}\alpha_{21}(Y) & a_{22}\alpha_{22}(Y) & \ldots & a_{2r}\alpha_{2r}(Y) \\ \vdots & \vdots & & \vdots \\ a_{r1}\alpha_{r1}(Y) & a_{r2}\alpha_{r2}(Y) & \ldots & a_{rr}\alpha_{rr}(Y) \end{vmatrix} \tag{6.7}$$

as follows: $m_i = m(\lambda_i)^{-1/n}$, where λ_i is some non-zero eigenvalue of π_y.

Now the problem reduces to a construction of operators E^{y}_{ij}. Operators E^{y}_{ii} are ordinary superspin-projection operators. The expressions of superspin-projection operators for a general superfield $\phi_i(x, \theta)$ are given in the appendix. Projection operators for scalar superfield $\phi(x, \theta)$ were given by Salam and Strathdee (1975), and for symmetrical tensor and tensor-bispinor superfields $\phi_{\mu_1\ldots\mu_s}(x, \theta)$, $\phi_{\alpha\mu_1\ldots\mu_s}(x, \theta)$ the projection operators were given by Sokatchev (1975). Using the second Casimir operator of supersymmetry algebra, it is possible to derive projection operators for a general superfield $\phi_i(x, \theta)$.

In the following section we illustrate the general schema in the case of superspin-$\frac{3}{2}$ equation for vector and scalar superfields. Superspin-$\frac{3}{2}$ contains Poincaré spin-2 and -$\frac{3}{2}$, and for that reason the given equation may be useful in supergravity.

7. Superspin-$\frac{3}{2}$ equation

In this section we demonstrate that superspin-$\frac{3}{2}$ must be described with the help of two superfields: vector superfield $\phi_1(x, \theta) = h^{\mu}(x, \theta)$ and scalar superfield $\phi_2(x, \theta) = \phi(x, \theta)$.

We shall firstly deal with the vector superfield $h^{\mu}(x, \theta)$. The superspin-$\frac{3}{2}$ projection operator, which we denote as $E^{3/2}_{11}$, is the following (A5)

$$(E^{3/2}_{11})^{\kappa}{}_{\lambda} \equiv (E^{3/2}_{1})^{\kappa}{}_{\lambda} = \tfrac{2}{3}[1 + (1/4\Box)(\bar{D}D)^2](\eta^{\kappa}{}_{\lambda} - \partial^{\kappa}\,\partial_{\lambda}/\Box) - (1/6\Box)\,\partial^{\rho}\varepsilon_{\rho\sigma}{}^{\kappa}{}_{\lambda}\bar{D}\mathrm{i}\gamma^{\sigma}\gamma^{5}D. \tag{7.1}$$

Since $E^{3/2}_{11}$ contains non-local terms $\Box^{-2}$ and we want to give a second-order equation, we must use other operators E^{y}_{11} to eliminate this non-local term. There are three possibilities: superspin-1, -$\frac{1}{2}$ or -0 projection operators. Similarly as in the case of symmetrical tensor $h^{\mu\nu}(x)$ we use the operators of lowest superspin-0. In analogy to Ogievetsky and Sokatchev (1966, 1977a), we take the operator $E^{0}_{0+} + E^{0}_{0-}$ (from the considerations of symmetry it is not possible to prefer one of the chiral components + or −) and denote it as E^0_{11}. From (A5)

$$(E^0_{11})^\kappa{}_\lambda = -(1/4\Box)(\bar{D}D)^2\,\partial^\kappa\,\partial_\lambda/\Box. \tag{7.2}$$

Now it is easy to see that the operator $E^{3/2}_{11} - \frac{2}{3}E^0_{11}$ has the required non-locality $\Box^{-1}$. From the considerations of the root method the same operator was derived by Ogievetsky and Sokatchev (1976, 1977a).

If we use only the vector superfield $h^\mu(x, \theta)$, we obtain the following equation (Ogievetsky and Sokatchev 1976, 1977a)

$$-\Box(E^{3/2}_{11} - \tfrac{2}{3}E^0_{11})^\kappa{}_\lambda h^\lambda(x, \theta) = m^2 h^\kappa(x, \theta). \tag{7.3}$$

Equation (7.3) describes superspin-$\frac{3}{2}$ with mass m and superspin-0 with non-physical mass $\mathrm{i}m(\frac{3}{2})^{1/2}$.

In order to eliminate superspin-0 or describe it with physical mass spectrum we introduce the scalar superfield $\phi(x, \theta)$ from which we extract superspin-0. The corresponding projection operator is (A7)

$$E^0_{22} = -(1/4\Box)(\bar{D}D)^2. \tag{7.4}$$

It remains to obtain operators E^0_{12} and E^0_{21} which satisfy (6.2). It is possible to verify that E^0_{12} and E^0_{21} are the following

$$(E^0_{12})^\kappa = \bar{D}D\,\partial^\kappa/2\Box \qquad (E^0_{21})_\lambda = -\bar{D}D\,\partial_\lambda/2\Box. \tag{7.5}$$

The general equation for $h^\mu(x, \theta)$ and $\phi(x, \theta)$ is the following from (6.3) (we take $a_{11} = 1$, $a_{21} = a$, $a_{12} = b$, $a_{22} = c$)

$$-\Box\begin{vmatrix} (E^{3/2}_{11} - \frac{2}{3}E^0_{11})^\kappa{}_\lambda & b(E^0_{12})^\kappa \\ a(E^0_{21})_\lambda & c(E^0_{22}) \end{vmatrix}\begin{vmatrix} h^\lambda(x, \theta) \\ \phi(x, \theta) \end{vmatrix} = m^2\begin{vmatrix} h^\kappa(x, \theta) \\ \phi(x, \theta) \end{vmatrix}. \tag{7.6}$$

Using the expressions of operators E^γ_{ij}, (7.6) may be written as

$$\begin{gathered}\tfrac{2}{3}[(\Box + \tfrac{1}{4}(\bar{D}D)^2)h^\kappa(x, \theta) - \partial^\kappa\,\partial_\lambda h^\lambda(x, \theta)] - \tfrac{1}{6}\partial^\rho\varepsilon_{\rho\sigma}{}^\kappa{}_\lambda\bar{D}\mathrm{i}\gamma^\sigma\gamma^5 D h^\lambda(x, \theta) \\ + \tfrac{1}{2}b\bar{D}D\,\partial^\kappa\phi(x, \theta) + m^2 h^\kappa(x, \theta) = 0, \\ -\tfrac{1}{2}a\bar{D}D\,\partial_\lambda h^\lambda(x, \theta) - \tfrac{1}{4}c(\bar{D}D)^2\phi(x, \theta) + m^2\phi(x, \theta) = 0.\end{gathered} \tag{7.7}$$

The equation obtained describes superspin-$\frac{3}{2}$ with mass m and depending on the choice of parameters a, b and c two, one or no superspins-0. The analysis of mass spectrum is the same, as in § 4. We decompose $\pi^{00} = \pi^{3/2} + \pi^0$

$$\pi^{3/2} = \begin{vmatrix} E^{3/2}_{11} & 0 \\ 0 & 0 \end{vmatrix}, \qquad \pi^0 = \begin{vmatrix} -\frac{2}{3}E^0_{11} & bE^0_{12} \\ aE^0_{21} & cE^0_{22} \end{vmatrix}, \tag{7.8}$$

and now the investigation of the mass spectrum reduces to the investigation of non-zero eigenvalues of matrices $\pi_{3/2}$ and π_0

$$\pi_{3/2} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \qquad \pi_0 = \begin{vmatrix} -\frac{2}{3} & b \\ a & c \end{vmatrix}. \tag{7.9}$$

The only non-zero eigenvalue of $\pi_{3/2}$ is equal to 1 and therefore the corresponding mass is equal to m. The eigenvalues of π_0 are calculated from

$$6\lambda_{1,2} = 3c - 2 \pm [(3c+2)^2 + 36ab]^{1/2}. \tag{7.10}$$

In order to have the physical mass spectrum, ab and c must satisfy

$$c \geq \tfrac{2}{3}, \qquad 3ab+2c \leq 0, \qquad (3c+2)^2+36ab \geq 0. \tag{7.11}$$

The physical region of parameters is determined by the line

$$3ab+2c=0, \tag{7.12}$$

where the eigenvalues are

$$\lambda_1 = c-\tfrac{2}{3}, \qquad \lambda_2 = 0, \tag{7.13}$$

and by the parabola

$$(3c+2)^2+36ab=0, \tag{7.14}$$

which gives coincident eigenvalues

$$\lambda_1=\lambda_2=\tfrac{1}{6}(3c-2). \tag{7.15}$$

The points on the line (7.12), therefore, in addition to superspin-$\frac{3}{2}$ describe one superspin-0 with mass $m\lambda_1^{-1/2}$. The points on the parabola may be regarded as unphysical since π_0 satisfies $(\pi_0-\lambda_1)^2=0$ and has for that reason only one eigenvector. The other points in the physical region determined by (7.11) give different eigenvalues $\lambda_1 \neq \lambda_2$, and describe two superspins-0.

Single superspin-$\frac{3}{2}$ is described by the parameters (π^0 is nilpotent)

$$ab=-\tfrac{4}{9}, \qquad c=\tfrac{2}{3}. \tag{7.16}$$

The inverse problem—how to find a superfield equation with given mass spectrum of superspin-0 states—is solved similarly as in § 4. From the masses we determine λ_1 and λ_2, and then the coefficients ab and c are calculated from

$$\begin{aligned} ab &= -\lambda_1\lambda_2-\tfrac{2}{3}(\lambda_1+\lambda_2)-\tfrac{4}{9}, \\ c &= \lambda_1+\lambda_2+\tfrac{2}{3}. \end{aligned} \tag{7.17}$$

As we have seen in this section, the general schema proposed in § 6 works similarly as in the ordinary field case. It also works in the case of known equations for scalar and bispinor superfields.

To conclude this section, it is interesting to note that in our case the projection operators E^0_{ij} had the following decomposition

$$E^0_{ij}=\mathscr{E}_{ij}P^0_{ij}, \tag{7.18}$$

where $\mathscr{E}_{ij}$ satisfy $\mathscr{E}_{ij}\mathscr{E}_{jk}=\mathscr{E}_{ik}$, and P^0_{ij} satisfy (2.8), i.e. both factors are projection operators.

$$\mathscr{E}_{11}=\mathscr{E}_{22}=-(1/4\square)(\bar{D}D)^2, \qquad \mathscr{E}_{12}=-\mathscr{E}_{21}=(1/2\sqrt{\square})\bar{D}D, \tag{7.19}$$

$$(P^0_{11})^\mu{}_\nu=\partial^\mu\partial_\nu/\square, \qquad (P^0_{12})^\mu=\partial^\mu/\sqrt{\square}, \qquad (P^0_{21})_\nu=\partial_\nu/\sqrt{\square}, \qquad P^0_{22}=1. \tag{7.20}$$

Operators (7.20) give the system of covariant spin-projection operators for vector and scalar fields. The well known Kemmer–Duffin spin-0 equation is, from (2.12), the following

$$\mathrm{i}\sqrt{\Box}\begin{vmatrix} 0 & (P_{12}^0)^\mu \\ (P_{21}^0)_\nu & 0 \end{vmatrix}\begin{vmatrix} h^\nu \\ \phi \end{vmatrix} = m\begin{vmatrix} h^\mu \\ \phi \end{vmatrix}.$$

It should be remarked that the decomposition (7.18) is always possible, as it is also seen from the general expressions of projection operators (A1), but the factors $\mathscr{C}_{ij}$ are not, in general, projection operators. In the chiral superfield case the factors $\mathscr{C}_{ij}$ are certainly projection operators, otherwise they are not.

8. Conclusions

In this paper the full description of all second-order equations for a symmetrical tensor field $h^{\mu\nu}$ describing spin-2 and -0 states has been given. The general principles which are used in the ordinary field case are also applicable in the superfield case. The general form of superfield equations of motion has been proposed. As an example, the superfield equation for vector and scalar superfields describing superspins-$\frac{3}{2}$ and -0 has been given. It is demonstrated that the derivation of equations for a symmetrical tensor field $h^{\mu\nu}$ is entirely analogous to the derivation of superfield equations for vector and scalar superfields h^μ and ϕ.

Here we outline once more the general procedure of how to write down an equation corresponding to a given mass spectrum, or to establish the particle content and masses of a given equation.

8.1. Equations with given mass spectrum

8.1.1. Spin-2 and -0. Spin-2 has mass m, choose the masses of spin-0 particles $m\lambda_1^{-1/2}$ and $m\lambda_2^{-1/2}$. Parameters ab and c are determined from (4.6), (4.9) or (4.10). Now we choose a, b and c, and the corresponding equation is given by (3.8).

8.1.2. Superspin-$\frac{3}{2}$ and -0. Superspin-$\frac{3}{2}$ has mass m, choose the masses of superspin-0 states $m\lambda_1^{-1/2}$ and $m\lambda_2^{-1/2}$. Parameters ab and c are determined from (7.13), (7.16) or (7.17). Now we choose a, b and c, and the corresponding equation is given by (7.7).

8.2. Masses and particle content of a given equation

8.2.1. Spins-2 and -0. Equation (3.9) is given. Parameters a, b and c are then determined from (3.10). Formula (4.3) gives the eigenvalues λ_1 and λ_2, which in turn determine the masses and particle content.

8.2.2. Superspin-$\frac{3}{2}$ and -0. Equation (7.7) is given. Formula (7.10) gives the eigenvalues λ_1 and λ_2, which in turn determine the masses and particle content.

Appendix. Superfield projection operators

We give the superspin-projection operators for a general superfield $\phi_i(x, \theta)$ which transforms under Lotentz transformations according to some irreducible representation (k, l). Poincaré spins s contained in the representation (k, l) are: $s =$

$k+l, k+l-1, \ldots, |k-l|$. To each Poincaré spin s corresponds four superspins $Y = s+\frac{1}{2}, s, s, s-\frac{1}{2}$ which are extracted with the help of projection operator E^Y (Loide 1984b)

$$\begin{aligned} E^{s+1/2} &= (2s+1)^{-1}[1+(1/4\Box)(\bar{D}D)^2][s+1-(1/4\Box)\,\partial^\kappa \varepsilon_{\kappa\mu\rho\sigma} S^{\rho\sigma} \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]P^s \\ E^s_+ &= -(1/8\Box)[(\bar{D}D)^2 + 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]P^s \\ E^s_- &= -(1/8\Box)[(\bar{D}D)^2 - 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]P^s \\ E^{s-1/2} &= (2s+1)^{-1}[1+(1/4\Box)(\bar{D}D)^2][s+(1/4\Box)\,\partial^\kappa \varepsilon_{\kappa\mu\rho\sigma} S^{\rho\sigma} \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]P^s. \end{aligned} \tag{A1}$$

Here D_α is the covariant spinor derivative (Salam and Strathdee 1975)

$$D_\alpha = \partial/\partial\bar{\theta}_\alpha - \tfrac{1}{2}\mathrm{i}(\not\partial\theta)_\alpha \tag{A2}$$

and P^s are the projection operators of Poincaré spins (in the terminology of § 2 $P^s \equiv P^s_{ii}$).

Projection operators for vector superfield $h^\mu(x, \theta)$ are obtained as follows. The generators $S^{\rho\sigma}$ are

$$(S^{\rho\sigma})^\kappa{}_\lambda = \eta^{\rho\kappa}\eta^\sigma{}_\lambda - \eta^{\sigma\kappa}\eta^\rho{}_\lambda \tag{A3}$$

and spin projection operators P^1, P^0

$$(P^1)^\kappa{}_\lambda = \eta^\kappa{}_\lambda - \partial^\kappa\,\partial_\lambda/\Box, \qquad (P^0)^\kappa{}_\lambda = \partial^\kappa\,\partial_\lambda/\Box. \tag{A4}$$

To the Poincaré spin-1 correspond superspins $Y = \frac{3}{2}$, 1, 1 and $\frac{1}{2}$ which are extracted with the help of projection operators

$$\begin{aligned} (E^{3/2}_1)^\kappa{}_\lambda &= \tfrac{2}{3}[1+(1/4\Box)(\bar{D}D)^2](\eta^\kappa{}_\lambda - \partial^\kappa\,\partial_\lambda/\Box) - (1/6\Box)\,\partial^\rho \varepsilon_{\rho\sigma}{}^\kappa{}_\lambda \bar{D}\mathrm{i}\gamma^\sigma\gamma^5 D \\ (E^1_{1+})^\kappa{}_\lambda &= -(1/8\Box)[(\bar{D}D)^2 + 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D](\eta^\kappa{}_\lambda - \partial^\kappa\,\partial_\lambda/\Box) \\ (E^1_{1-})^\kappa{}_\lambda &= -(1/8\Box)[(\bar{D}D)^2 - 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D](\eta^\kappa{}_\lambda - \partial^\kappa\,\partial_\lambda/\Box) \\ (E^{1/2}_1)^\kappa{}_\lambda &= \tfrac{1}{3}[1+(1/4\Box)(\bar{D}D)^2](\eta^\kappa{}_\lambda - \partial^\kappa\,\partial_\lambda/\Box) + (1/6\Box)\,\partial^\rho \varepsilon_{\rho\sigma}{}^\kappa{}_\lambda \bar{D}\mathrm{i}\gamma^\sigma\gamma^5 D. \end{aligned} \tag{A5}$$

To the Poincaré spin-0 correspond $Y = \frac{1}{2}$, 0, 0

$$\begin{aligned} (E^{1/2}_0)^\kappa{}_\lambda &= [1+(1/4\Box)(\bar{D}D)^2]\,\partial^\kappa\,\partial_\lambda/\Box \\ (E^0_{0+})^\kappa{}_\lambda &= -(1/8\Box)[(\bar{D}D)^2 + 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]\,\partial^\kappa\,\partial_\lambda/\Box \\ (E^0_{0-})^\kappa{}_\lambda &= -(1/8\Box)[(\bar{D}D)^2 - 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]\,\partial^\kappa\,\partial_\lambda/\Box. \end{aligned} \tag{A6}$$

Here the lower indices denote the corresponding Poincaré spins, + and − distinguish the corresponding chiral components.

In the case of scalar superfield $\phi(x, \theta)$ the projection operators are (Salam and Strathdee 1975)

$$\begin{aligned} E^{1/2} &= 1+(1/4\Box)(\bar{D}D)^2 \\ E^0_+ &= -(1/8\Box)[(\bar{D}D)^2 + 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D] \\ E^0_- &= -(1/8\Box)[(\bar{D}D)^2 - 2\mathrm{i}\,\partial_\mu \bar{D}\mathrm{i}\gamma^\mu\gamma^5 D]. \end{aligned} \tag{A7}$$

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